

Some notes on numerical optimal control

Numerical determination of optimal trajectories

1.1 Two point boundary value problem

In previous chapter variational techniques were used to derive necessary conditions for optimal control. The problem was stated as follows:

Consider the problem of minimizing the performance measure

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (1.1)$$

subject to:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (1.2)$$

The problem is to find an admissible control $\mathbf{u}^*(t)$ that causes the system (1.2) to follow an admissible trajectory $\mathbf{x}^*(t)$ that minimizes the performance measure (1.1).

The *hamiltonian*, has been defined as:

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda^T(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

Assuming that the state and control variables are not constrained, that the final time t_f is fixed and the final state is free, we can summarize the two-point boundary value problem (TPBVP) that results from the variational approach by the equations, (Kirk, 2004):

$$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \lambda} = \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*, t) \quad (1.3)$$

$$\dot{\lambda}^* = -\frac{\partial H}{\partial \mathbf{x}} \quad (1.4)$$

$$0 = \frac{\partial H}{\partial \mathbf{u}} \quad (1.5)$$

$$\mathbf{x}^*(t_0) = \mathbf{x}_0 \quad (1.6)$$

$$\lambda^*(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) \quad (1.7)$$

From these five sets of conditions it is desired to obtain an explicit relationship for $\mathbf{x}^*(t)$ and $\mathbf{u}^*(t)$, $t \in [t_0, t_f]$. The difficulty of solving the differential equations in this case is caused by the combination of split boundary values and nonlinear differential equations.

Note that the differential equations (1.3) and (1.4) should be solved, in general, simultaneously because they depend on the same variables (the states and the adjoint variables). The initial or final conditions, however, are split as shown in (1.6) and (1.7), as the first ones are the values of the states at the initial time and the rest are the values of the adjoint variables at the final time. A classical numerical method for integration cannot be applied.

The algorithm (Kirk, 2004) is based on the observation that $u^*(t)$ which minimizes the hamiltonian will minimize J . Thus, the numerical algorithm will determine the control u^* which makes the first derivative of the hamiltonian equal to zero.

- Write the hamiltonian $H(x(t), u(t), \lambda(t))$.

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda^T \cdot \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (1.8)$$

Calculate the first derivative of the hamiltonian with respect to u : $\partial H / \partial u$.

- Write the differential equations of the adjoint variables:

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i},$$

- Determine the final conditions for $\lambda(t_f)$
- Start with a random discrete approximation for $u(t)$, $t \in [0, t_f]$. For example divide the interval $[t_0, t_f]$ into N subintervals and consider the control u constant on each subinterval.

$$u(t) = u(t_k), \quad k = 1..N$$

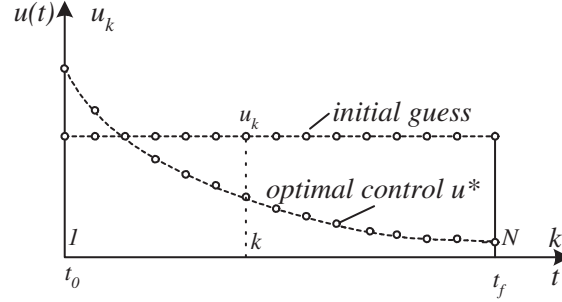


Figure 1.1: Discrete control variable

- Minimize $H(\mathbf{u})$ using a Matlab function or a numerical algorithm to determine the minimum (for example gradient methods). The function to be minimized has to solve at each step:
 - Using the vector u integrate the state equations (use Runge-Kutta or Euler methods), with the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ to obtain a state trajectory constant on N subintervals.
 - Calculate $\lambda(t_f)$
 - Solve the adjoint system of differential equations reversely, from t_f to t_0 .
 - Using u , x and λ calculate the first derivative of the hamiltonian.
 - Return the norm of the Hamiltonian or the norm of the first derivative of the Hamiltonian.

The procedure to solve the optimal control problem using the steepest descent method can be summarized as follows:

- Select a discrete approximation of the control variable $\mathbf{u}(t)$, $t = [t_0, t_f]$, by subdividing the interval $[t_0, t_f]$ into N subintervals and considering the control variable as being piecewise constant during each of these subintervals.
- Using the N control values, integrate the state equations (1.3) from t_0 to t_f , with initial conditions (1.6) and obtain a state trajectory as a piecewise constant vector function.
- Calculate $\lambda(t_f)$ by substituting $\mathbf{x}(t_f)$ into (1.7).
- Integrate (1.4) from t_f to t_0 , with final conditions (1.7) using the piecewise-constant values of the states calculated before.

- If

$$\left\| \frac{\partial H}{\partial \mathbf{u}} \right\| \leq \varepsilon \quad (1.9)$$

where ε is a small positive constant, terminate the iterative procedure and output the extremal state and control values.

- If the stopping criterion (1.9) is not satisfied, generate a new piecewise constant control, function given by:

$$\mathbf{u} \leftarrow \mathbf{u} - \tau \frac{\partial H}{\partial \mathbf{u}} \quad (1.10)$$

where τ is the step of the steepest descent method.

Bibliography

Kirk, D. E. (2004). *Optimal Control Theory. An Introduction*. Dover Publications, Inc.